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ON FUZZY GENERALIZED PREOPEN SETS

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Abstract: We aim to show in this paper that fuzzy μ-preopen sets of a GFTS or fuzzy μ-space X may be equivalent to fuzzy μ-open in X. Moreover we portray generalized fuzzy paracompactness of a fuzzy μ-space X via fuzzy μ-preopen sets in X.

Keywords and Phrases: Fuzzy μ-preopen set, fuzzy μ-dense, fuzzy μ-locally finite, fuzzy μ-paracompact.

2020 Mathematics Subject Classification: 54A05, 54D20.

1. Introduction

Chang [3] presented fuzzy topology after the discovery of fuzzy sets by Zadeh [14]. The notion of generalized topology proposed by Csaszar in [6]. Let I^X denotes non empty set X. A fuzzy subcollection μ of I^X is called a generalized fuzzy topology [4] on X if $0_X \in \mu$ and $\bigvee\{\xi_\alpha \alpha \in \Delta\} \in \mu$ whenever $\xi_\alpha \in \Delta$ for every $\alpha \in \mu$. The terms FS, Fμ-O,Fμ-PO and GFTS stands for fuzzy set, fuzzy μ-open, fuzzy μ-preopen and generalized fuzzy topological space respectively. A fuzzy set $\xi \in \mu$ is called fuzzy μ-open [4] of (X, μ). The complement of fuzzy μ-open set is called fuzzy μ-closed of (X, μ). A FS λ of X, the intersection of all fuzzy μ-closed sets containing λ is the generalized fuzzy closure of λ and is denoted by $c_\mu(\xi)$. Also for a fuzzy set ξ of X, the union of all fuzzy μ-open sets contained in ξ is the generalized fuzzy interior of ξ and is denoted by $i_\mu(\xi)$. For our better understanding, a FS ξ of X is μ-open (resp. fuzzy μ-closed) if and only if $\xi = i_\mu(\xi)$ (resp. fuzzy $\xi = c_\mu(\xi)$). Also for a FS ξ of X, we have $c_\mu(\xi) = 1_X - i_\mu(\xi)$ [4].

A fuzzy set ξ of X is called fuzzy μ-preopen [5] if $\xi \subset i_\mu(c_\mu(\xi))$. The complement of a fuzzy μ-preopen set is called fuzzy μ-preclosed. Hence a FS ξ of X is fuzzy μ-preclosed if $c_\mu(i_\mu(\xi)) \subset \xi$. Analogouly a FS ξ of X is fuzzy μ-preopen iff there exists a fuzzy μ-open set β such that $\xi \subset \beta \subset c_\mu(\xi)$. Related this paper recent studies have been found in [1], [2], [9] and [11]. Swaminathan and Sheela [13] explored a covering property namely fuzzy μ-precompactness via fuzzy μ-preopen sets. In this article, we obtain some more properties of fuzzy μ-preopen sets on GFTS or fuzzy μ-spaces.

2. Fuzzy μ-preopen Sets

We now introduce the main results:

Definition 2.1. *A FS γ of X is said to be fuzzy μ-dense if $c_\mu(\gamma) = 1_X$.*

Every Fμ-O sets in X are Fμ-PO in X but the converse need not be true. This will lead us to know that when Fμ-PO sets in X may be Fμ-O in X. In light of this, we shall first instigate the following.

Definition 2.2. *A GFTS or fuzzy μ-space X is said to be a fuzzy μ-door space if every fuzzy subset of X is either Fμ-O or fuzzy μ-closed.*

Theorem 2.1. *Each Fμ-PO set of a fuzzy μ-door space is Fμ-O.*

Proof. Let ξ be a Fμ-PO set of a fuzzy μ-door space X. Now $\xi \subset i_\mu(c_\mu(\xi))$. Suppose ξ is not Fμ-O, then it is fuzzy μ-closed. Then $c_\mu(\xi) = \xi$ gives $i_\mu(c_\mu(\xi)) = i_\mu(\xi)$. By the definition, $i_\mu(\xi) = i_\mu(c_\mu(\xi)) \subset \xi$. Since ξ is not Fμ-O, $i_\mu(\xi) = i_\mu(c_\mu(\xi)) \subsetneq \xi$ which constitutes a contradiction that $\xi \subset i_\mu(\xi) = i_\mu(c_\mu(\xi))$.

Theorem 2.2. *Each singleton in X is either Fμ-O or fuzzy μ-closed if each Fμ-PO set of X is Fμ-O.*

Proof. Assume that for $x_\alpha \in X, \{x_\alpha\}$ is not fuzzy μ-closed. Then for each Fμ-PO set in X is Fμ-O, $\{x_\alpha\}$ is not fuzzy μ-preclosed and so

$$c_\mu(i_\mu(\{x_\alpha\})) \not\subseteq \{x_\alpha\}. \tag{2.1}$$

Case i: $\{x_\alpha\}$ is Fμ-O. Then $c_\mu(i_\mu(\{x_\alpha\})) = c_\mu(\{x_\alpha\})$ which contradicts (2.1).
Case ii.$\{x_\alpha\}$ is not Fμ-O. Then $c_\mu(i_\mu(\{x_\alpha\})) = c_\mu(0_X) = 0_X$ which contradicts (2.1).

Therefore $\{x_\alpha\}$ is Fμ-O.

Remark 2.3. *In a GFTS, Theorem 2.2 may not be true since in a GFTS, X may not be Fμ-O and 0_X may not be fuzzy μ-closed, This follows from the following Example.*

Example 2.4. Let $X = \{a, b, c, d\}$. Then $\gamma_1 = \frac{0}{a} + \frac{0}{b} + \frac{1}{c} + \frac{0}{d}$ and $\gamma_2 = \frac{0}{a} + \frac{1}{b} + \frac{1}{c} + \frac{0}{d}$ are two fuzzy sets: Let $\mu = \{0_X, \gamma_1, \gamma_2\}$ be a generalized fuzzy topology. Now each Fμ-PO set is Fμ-O but $\gamma_3 = \frac{1}{a} + \frac{0}{b} + \frac{0}{c} + \frac{0}{d}$ is neither Fμ-O nor fuzzy μ-closed.

Theorem 2.5. *Each fuzzy μ-dense set in X is Fμ-O if each Fμ-PO set in X is Fμ-O.*

Proof. Let λ be fuzzy μ-dense in X. Then $i_\mu(c_\mu(\lambda)) = 1_X$, as X is Fμ-O in X. Clearly $\lambda \subset i_\mu(c_\mu(\lambda))$. Hence λ is Fμ-PO. From assumption, we conclude that λ is Fμ-O.

In Example 2.4, each Fμ-PO set $(\digamma, \mu)$ is Fμ-O and $\gamma_4 = \frac{0}{a} + \frac{0}{b} + \frac{1}{c} + \frac{1}{d}$ is fuzzy μ-dense. But γ_4 is not Fμ-O in X. So we deduce that Theorem 2.5 need not be true in a GFTS.

A FS λ in a GFTS X is purely Fμ-PO if λ is Fμ-PO but not Fμ-O. Hence if λ is purely Fμ-PO in X, then $\lambda \neq 0_X, 1_X$. It follows that X is fuzzy μ-dense in a GFTS.

Theorem 2.6. *There exists a fuzzy μ-dense set other than X in a GFTS X if X contains a purely Fμ-PO set.*

Proof. Consider a Fμ-PO set λ in a GFTS X. Now $\lambda \subset i_\mu(c_\mu(\lambda)) \subset c_\mu(\lambda)$. If $c_\mu(\lambda) \neq 1_X$, then $\lambda \neq 1_X$ is fuzzy μ-dense in X. If $c_\mu(\lambda) \neq 1_X$, replace $\beta = i_\mu(c_\mu(\lambda))$. Observe that $\beta \neq 1_X$ and $c_\mu(\lambda) = c_\mu(\beta)$. Now

$$c_\mu((1_X - \beta) \cup \lambda) \supset c_\mu(1_X - \beta) \cup c_\mu(\lambda) = (1_X - \beta) \cup c_\mu(\beta) \supset (1_X - \beta) \cup \beta = 1_X.$$

Therefore we have $c_\mu((1_X - \beta) \cup \lambda) = 1_X$. Since λ is purely Fμ-PO and $\lambda \subset \beta, \beta - \lambda \neq 0_X$. Consequently, it can be inferred that $(1_X - \beta) \cup \lambda \neq 1_X$. Therefore we get $(1_X - \beta) \cup \lambda \neq 1_X$ is fuzzy μ-dense in X.

Corollary 2.7. *Let λ be Fμ-PO in a GFTS X and there exist $\beta \subset X$ such that $\lambda \subset \beta \subset c_\mu(\lambda)$. Then $(1_X - \beta) \cup \lambda$ is fuzzy μ-dense in X.*

Proof. Analogous to that of Theorem 2.6.

Theorem 2.8. *Each FS in GFTS X is Fμ-PO iff each Fμ-O set in X is fuzzy μ-closed.*

Proof. Consider each FS of X is Fμ-PO and ζ is Fμ-O in X. So $1_X - \zeta$ is fuzzy

μ-closed. By assumption, $c_\mu(1_X - \zeta)$ is Fμ-PO, we have

$$c_\mu(1_X-\zeta) \subset i_\mu(c_\mu(c_\mu(1_X-\zeta))) = i_\mu(c_\mu(1_X-\zeta)) \text{ since } 1_X-\zeta = c_\mu(1_X-\zeta) = i_\mu(1_X-\zeta).$$

So, we see that $1_X - \zeta = c_\mu(1_X - \zeta) \subset i_\mu(1_X - \zeta)$. By the definition, $i_\mu(1_X - \zeta) \subset 1_X - \zeta$.
Thus, we have $1_X - \zeta = i_\mu(1_X - \zeta)$ and which implies $1_X - \zeta$ is Fμ-O and hence ζ is fuzzy μ-closed.

Conversely, let each Fμ-O set in X is fuzzy μ-closed and λ be any fuzzy subset of X. Then $1_X - c_\mu(\lambda)$ is Fμ-O and also fuzzy μ-closed. Hence $1_X - c_\mu(\lambda) = c_\mu(1_X - c_\mu(\lambda)) = 1_X - i_\mu(c_\mu(\lambda))$ which implies $c_\mu(\lambda) = i_\mu(c_\mu(\lambda))$. Therefore $\lambda \subset c_\mu(\lambda) = i_\mu(c_\mu(\lambda))$. Thus λ is Fμ-PO.

Theorem 2.9. *Each subset $\lambda(\neq 0_X, 1_X)$ of GFTS X is Fμ-PO iff each Fμ-O set $\zeta(\neq 0_X, 1_X)$ in X is fuzzy μ-closed.*
Proof. Analogous to that of Theorem 2.8.

3. Fuzzy Generalized Paracompactness via Fuzzy μ-preopen Sets

A Fμ-O cover of X is defined to be the collection $\mathscr{M}$ of Fμ-O sets of X such that $\bigcup_{M\in\mathscr{M}} M = 1_X$. Now in GFTS, X may not be Fμ-O and so even the union of all Fμ-O sets of X may not be equal to 1_X. Now definition of Fμ-O covers become invalid. In order to avoid such invalidity, we extend the existing ideas of covering properties in GT spaces slightly.

Assume that $X_\mu = \bigcup_{V\in\mu} V$. Obviously, X_μ is Fμ-O. If X is a GFTS with $X \notin \mu$, then X_μ is not fuzzy μ-closed. Emphasize in the event of fuzzy μ-spaces, $X_\mu = 1_X$. A collection $\mathscr{M}$ of fuzzy subsets of X is called a fuzzy cover of X if $\bigcup_{V\in\mathscr{M}} V = X_\mu$. We write 'F$\mu$-O collection of X'and 'Fμ-PO collection of X to mean a collection consisting Fμ-O sets and Fμ-PO sets respectively of X. A Fμ-O collection (resp. Fμ-PO collection) $\mathscr{M}$ of X is said to be a Fμ-O (resp. Fμ-PO) cover of X if $\bigcup_{V\in\mathscr{M}} V = X_\mu$.

Definition 3.1. [10] *A fuzzy cover $\mathscr{A}$ of X is a fuzzy refinement of the fuzzy cover $\mathscr{B}$ of X if $\forall\ \alpha \in \mathscr{A}$, $\exists \beta \in \mathscr{A}$ such that $\alpha < \beta$.*

Definition 3.2. [10] *A collection $\mathscr{G}$ of fuzzy subsets of X is called fuzzy locally finite if for each $x_\alpha \in X$ there exists a fuzzy open set δ with $x_\alpha \in \delta$ meeting only finitely many members of $\mathscr{G}$.*

However, if $\mathscr{H}$ is a Fμ-PO cover and $\mathscr{G}$ is a Fμ-O cover of X, then $\mathscr{H}$ is called a Fμ-PO refinement of $\mathscr{G}$.

Definition 3.3. [12] *Let $\mathscr{A}$ and $\mathscr{B}$ be two fuzzy covers of a GFTS X. $\mathscr{A}$ is said to be a fuzzy s-refinement of $\mathscr{B}$ if for each $A \in \mathscr{A}$ there is a $B \in \mathscr{B}$ such that $A \subsetneq B$. An s-refinement $\mathscr{A}$ of $\mathscr{B}$ is said to be a fuzzy open s-refinement of $\mathscr{B}$ if all members of $\mathscr{B}$ are fuzzy open.*

Definition 3.4. *A collection $\mathscr{G}$ of fuzzy subsets of X is called fuzzy μ-locally finite if for each $x_\alpha \in X_\mu$, there exists a Fμ-O set δ with $x_\alpha \in \delta$ meeting only finitely many members of $\mathscr{G}$.*

Definition 3.5. *A space X is called a fuzzy μ-paracompact space if each Fμ-O cover of X has a fuzzy μ-locally finite Fμ-O refinement.*

Definition 3.6. *A Fμ-PO set β in a space X is said to be capped by a Fμ-O set if $\beta \subset \delta$ and $\beta \subset \eta$ for Fμ-O sets δ, η in X, then there exists a Fμ-O set γ such that $\beta \subset \gamma \subset \delta \cap \eta$.*

Example 3.1. Let $U = [0, +\infty)$ and for each $x \in U$.

$$\text{Let } \Omega_x(y) = \begin{cases} e^{-y} & if \quad 0 < y < x \\ 0 & if \quad y \geq x, \end{cases}$$

$$\Omega_\infty(y) = e^{-y}, \forall y \in U,$$

$$\mu = \{0, \Omega_\infty\} \cup \{\Omega_x, \forall x \in U\},$$

then it can be easily verified that (X, μ) is a GFTS. It is observed that Fμ-PO sets are always contained in a Fμ-O set. Consider two fuzzy sets

$$\Omega_w(y) = \begin{cases} e^{-w} & if \quad 0 < w < y < x, \\ 0 & if \quad y \geq x, \end{cases} \qquad \Omega_z(y) = \begin{cases} e^{-z} & if \quad 0 < w < z <, y < x, \\ 0 & if \quad y \geq x. \end{cases}$$

If a Fμ-PO set $\Omega_k(y), \forall k \in U$ is contained in the Fμ-O sets $\Omega_w(y), \Omega_z(y)$, then $\Omega_k(y)$ is capped by the Fμ-O set $\Omega_z(y)$. Therefore it is clear that all Fμ-PO sets in X are capped by a Fμ-O set in X.

Theorem 3.2. *If a Fμ-PO collection $\mathscr{M} = \{M_\beta | \alpha \in \Delta\}$ of X is fuzzy μ-locally finite, then there exists a Fμ-O collection $\mathscr{G} = \{G_\alpha | M_\beta \subset G_\alpha, \alpha \in \Delta\}$ such that $\mathscr{G}$ and $\mathscr{H} = \{c_\mu(G) | G \in \mathscr{G}\}$ both are fuzzy μ-locally finite.*

Then if $\mathscr{M}$ is a fuzzy cover of X, then $\mathscr{G}$ and $\mathscr{H}$ both are also fuzzy covers of 1_X.

Proof. The collection $\mathscr{M}$ of Fμ-PO sets is fuzzy μ-locally finite. Then for each

$x_\alpha \in X_\mu$, there exists a Fμ-O set N_{x_α} such that $M_\beta \cap N_{x_\alpha} \neq 0_X$ for finitely many $\beta \in \Delta$. Let $M_{\beta_k} \cap N_{x_\alpha} \neq 0_X$ for $\beta_k \in \Delta, k \in \{1, 2, \cdots, n\}, n \in \mathbb{N}$. Then $M_\beta \cap N_{x_\alpha} = 0_X$ for $\beta \in \Delta - \{\beta_1, \beta_2, \cdots, \beta_n\}$ which implies $c_\mu(M_\beta) \cap N_{x_\alpha} = 0_X$ for $\beta \in \Delta - \{\beta_1, \beta_2, \cdots, \beta_n\}$. Now for each $\beta \in \Delta$, there exists a Fμ-O set G_β such that $M_\beta \subset G_\beta \subset c_\mu(M_\beta)$. It gives $c_\mu(M_\beta) = c_\mu(G_\beta)$. So $c_\mu(G_\beta) \cap N_{x_\alpha} = 0_X$ for $\beta \in \Delta - \{\beta_1, \beta_2, \cdots, \beta_n\}$. Hence we see that N_{x_α} may intersect only finitely many members of $\mathscr{H} = \{c_\mu(G_\alpha) | \alpha \in \Delta\}$. Thus $\mathscr{H}$ is fuzzy μ-locally finite.

Since for each $\alpha \in \Delta, G_\alpha \subset c_\mu(G_\alpha)$ and the collection $\mathscr{H} = \{c_\mu(G_\alpha) | \alpha \in \Delta\}$ is fuzzy μ-locally finite, the collection $\{G_\alpha | \alpha \in \Delta\}$ is also fuzzy μ-locally finite.

Consider $\mathscr{M}$ is a fuzzy cover of X. Then for each $x_\alpha \in X_\mu$, there exists a $M_\beta \in \mathscr{M}$ such that $x_\alpha \in M_\beta$. Hence $x_\alpha \in M_\beta \subset G_\alpha \subset c_\mu(M_\beta) = c_\mu(G_\alpha)$. Hence it follows that $\mathscr{G}$ and $\mathscr{H}$ both are fuzzy covers of X.

Theorem 3.3. *Let Fμ-PO sets are capped by Fμ-O sets in X. Then the fuzzy space X is fuzzy μ-paracompact if and only if each Fμ-O covers of X has a fuzzy μ-locally finite Fμ-PO refinement.*

Proof. Since Fμ-O sets on a space X are also Fμ-PO in X. Necessity holds.
To prove the sufficiency, let $\mathscr{M} = \{M_\alpha | \alpha \in \Delta\}$ be a Fμ-O cover of X and $\mathscr{H} = \{S_\beta |\ \beta \in \Gamma\}$ be a fuzzy μ-locally finite μ-preopen refinement of $\mathscr{M}$. By Theorem 3.2, there exists a fuzzy μ-locally finite Fμ-O cover $\mathscr{G} = \{G_\beta | G_\beta \subset S_\beta, \in \Gamma\}$ and for each $x_\alpha \in X_\mu$, there exists a $\beta(x_\alpha) \in \Gamma$ such that $x_\alpha \in S_{\beta(x_\alpha)} \subset U_{\beta(x_\alpha)}$. But for $\beta(x_\alpha) \in \Gamma$, there is a $\gamma(x_\alpha) \in \Delta$ such that $x_\alpha \in S_{\beta(x_\alpha)} \subset M_{\lambda(x_\alpha)}$. Since F$\mu$-PO sets in X are capped by Fμ-O sets, there exists a Fμ-O set Q_{x_α} such that $x_\alpha \in S_{\beta(x_\alpha)} \subset Q_{x_\alpha} \subset G_{\beta(x_\alpha)} \cap M_{\lambda(x_\alpha)} \subset M_{\lambda(x_\alpha)}$. So $\mathscr{Q} = \{Q_{x_\alpha} | x_\alpha \in X_\mu\}$ is a Fμ-O refinement of $\mathscr{M}$. Since $\mathscr{G}$ is fuzzy μ-locally finite and observe that $Q_{x_\alpha} \subset G_{\beta(x_\alpha)}$, $\mathscr{Q}$ is also fuzzy μ-locally finite. Hence we get a fuzzy μ-locally finite Fμ-O refinement $\mathscr{Q} = \{Q_{x_\alpha} | x_\alpha \in X_\mu\}$ of $\mathscr{M}$. Therefore X is fuzzy μ-paracompact.

4. Conclusion

In this paper we have studied and investigated fuzzy μ-preopen sets of a GFTS or fuzzy μ-space X and which may be equivalent to fuzzy μ-open in X. Further we extended our results to covering properties that is generalized fuzzy paracompactness of a fuzzy μ-space X via fuzzy μ-preopen sets in X. The notions of ideas can further be investigated in various covering and connectedness as well as in many topological properties of the literature.

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